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Quantization of a particle in the field of an elliptic flux tube

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Abstract. The role of the Floquet parameter in the quantization of a particle in the field of two parallel string-like solenoids is clarified.

1. Introduction

The problem of anyon statistics [1, 2] and related aspects of two-dimensional angular momentum has led to very detailed investigations of the quantization of particles in the magnetic field of a solenoid [3]. This seemingly simple problem is also of considerable interest for comparison with the behaviour of particles in the vicinity of field vortices [4]—and the latter are well known to allow some otherwise unusual quantum numbers. Although the one-solenoid problem seems relatively simple, it led to some false conclusions [1] concerning the quantization of two-dimensional angular momentum. The latter have been clarified and corrected in [3] where it is pointed out that a clear distinction between orbital and canonical angular momentum is absolutely essential to the argument. More recently the less symmetric problem of a particle in the field of two parallel string-like solenoids has been considered [5, 6], and again the question of how the angular momentum is quantized arises. The angular momentum and the related conserved quantity have, however, not been considered explicitly in these investigations, which therefore leave some crucial points unclarified.

Almost every investigation of the solenoid problem so far resorts at some point to the singular gauge transformation which—essentially—removes the electromagnetic field from the Hamiltonian, thus permitting an easier comparison with the case of the free particle. The unavoidable branch cut or Dirac string that one obtains in return is evidence of the fact that the field cannot really be removed. In order to circumvent this mathematical aspect the authors of [3] introduced a regularized—and hence nonsingular—version of this gauge transformation which has the advantage of demonstrating explicitly the rotational noncovariance of the resulting electromagnetic field which then makes it very plausible to accept a multivaluedness of the corresponding Schrödinger wavefunction.

In the problem of two parallel solenoids the lines of constant electromagnetic vector potential $|\boldsymbol{A}|$ are elliptic which suggests the use of elliptic cylinder coordinates. The field-free Hamiltonian then separates into a (periodic) Mathieu equation and a (nonperiodic)

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modified Mathieu equation. The wavefunctions of a periodic problem are well known to obey the Bloch condition ($v(\theta) = \mathrm{e}^{\mathrm{i}\nu\pi} \cdot v(\theta + \pi)$) which involves the Floquet parameter ν. We ask: What is the significance of ν in the problem of two solenoids and how is it related to the conserved angular momentum? In [5] ν is taken to be integral, whereas in [6] ν is stated to be nonintegral.

In the following we first derive the conserved quantity—the angular momentum which is equal to the orbital angular momentum plus a constant proportional to the magnetic flux—and then consider gauge transformations and the diagonalization of angular momentum and energy operators. We show that ν is an integer which quantizes the angular momentum.

2. Elliptic field geodesics and angular momentum

We consider the arrangement of [5] with the two parallel flux lines of infinitely long solenoids parallel to the z-axis and through the points $x = -a$ and $+a$. The vector potential $\boldsymbol{A}$ at a point (x, y) expressed in terms of polar coordinates (ρ, ϕ) with respect to these points as poles is given by

$$\boldsymbol{A} = \frac{\Phi}{2\pi}\left[\frac{\boldsymbol{e}_{\phi_1}}{\rho_1} + \frac{\boldsymbol{e}_{\phi_2}}{\rho_2}\right] \tag{1}$$

where Φ is the flux of the flux lines and $\boldsymbol{e}_{\phi_1}, \boldsymbol{e}_{\phi_2}$ are unit vectors in the ϕ_1, ϕ_2 directions. We take (r, φ) as polar coordinates with respect to the origin as a pole. In the following it is also very convenient to introduce elliptic coordinates μ and θ of the point (x, y) defined by

$$x = a\cosh\mu\cos\theta \qquad y = a\sinh\mu\sin\theta$$

where $0 \leqslant \mu < \infty, -\pi \leqslant \theta < \pi$. The unit vectors along the μ and θ directions are written $\boldsymbol{e}_\mu, \boldsymbol{e}_\theta$. The Lagrangian of a particle of mass m moving in the field of the solenoids is given by

$$\mathcal{L} = \frac{1}{2}m\dot{\boldsymbol{r}}^2 + \frac{e}{c}\dot{\boldsymbol{r}} \cdot \boldsymbol{A}(\boldsymbol{r}) \tag{2}$$

where $\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\varphi}\boldsymbol{e}_\varphi$. We set

$$\begin{aligned} R_\pm : &= (x \pm a)^2 + y^2 = a^2(\cosh\mu \pm \cos\theta)^2 \\ U_\pm : &= x(x \pm a) + y^2 \\ &= a^2\{\sinh^2\mu\sin^2\theta + \cosh\mu\cos\theta(\cosh\mu\cos\theta \pm 1)\}. \end{aligned} \tag{3}$$

Then

$$\boldsymbol{A} = \frac{\Phi}{2\pi}\left[a\left[\frac{1}{R_+} - \frac{1}{R_-}\right]\boldsymbol{e}_r + \frac{1}{r}\left[\frac{U_-}{R_-} + \frac{U_+}{R_+}\right]\boldsymbol{e}_\varphi\right] \tag{4}$$

and one can check that $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ (Coulomb gauge). Thus $\mathcal{L}$ with this value of $\boldsymbol{A}$ is not rotationally invariant. In fact, subjecting $\mathcal{L}$ to the rotational variation

$$\delta r_i = -\omega\epsilon_{ij}r_j \qquad \delta\varphi = \omega = \text{fixed}$$

one finds

$$\delta\mathcal{L} = a\omega\frac{e}{c}\frac{\Phi}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{x-a}{R_-} - \frac{x+a}{R_+}\right]. \tag{5}$$

Thus, the Lagrangian changes by a total time derivative and only the action is invariant. This rotational invariance-violating part, i.e. (5), however, contributes to the conserved quantity. Returning to $\mathcal{L}$ we see that

$$P_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mr^2\dot{\varphi} + \frac{e}{c} r \boldsymbol{e}_\varphi \cdot \boldsymbol{A}(\boldsymbol{r}) = mr^2\dot{\varphi} + \frac{e}{c}\frac{\Phi}{2\pi}\left[\frac{U_+}{R_+} + \frac{U_-}{R_-}\right] \tag{6}$$

and

$$(\boldsymbol{r} \times \boldsymbol{p}) \cdot \boldsymbol{e}_z = p_\varphi.$$

Since (with the help of the equation of motion)

$$\delta \mathcal{L}(\boldsymbol{r}, \dot{\boldsymbol{r}}) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{r}}} \cdot \delta \boldsymbol{r}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \delta\varphi\right] \tag{7}$$

we have

$$\omega \frac{\mathrm{d}}{\mathrm{d}t} M = 0$$

with

$$M = p_\varphi - a\frac{e}{c}\frac{\Phi}{2\pi}\left[\frac{x-a}{R_-} - \frac{x+a}{R_+}\right] \tag{8}$$

as the conserved quantity. Inserting the value of p_φ we obtain

$$M = mr^2\dot{\varphi} + 2\alpha \tag{9}$$

where $2\alpha = \frac{e\Phi}{c\pi}$. We observe that this expression is independent of a and hence of the ellipticity. It is important to distinguish here where we have a velocity-dependent potential between the *kinematical* or *orbital* angular momentum $mr^2\dot{\varphi}$ and the canonical angular momentum $\boldsymbol{r} \times \boldsymbol{p}$. In quantum mechanics it is $\boldsymbol{p}$ which becomes $-\mathrm{i}\hbar\boldsymbol{\nabla}$, not $m\dot{\boldsymbol{r}}$. Hence, the canonical angular momentum is not $mr^2\dot{\varphi}$. Thus, despite the rotational noninvariance of the original $\mathcal{L}$ but invariance of $\mathcal{L}$ augmented by the total time derivative to give the invariant expression, we have a conserved angular momentum M, and this is irrespective of whether the flux lines are present or not.

The lines of constant $\boldsymbol{A}$ can be seen to be elliptic geodesics in the (x, y)-plane with the flux lines passing through the foci. This clearly suggests the use of elliptic coordinates μ and θ. It is convenient to define the following quantities:

$$\begin{aligned} h(\theta,\mu) &:= \cosh 2\mu - \cos 2\theta \\ f(\theta,\mu) &:= \frac{1}{h}\sinh 2\mu \\ g(\theta,\mu) &:= \frac{1}{h}\sin 2\theta. \end{aligned} \tag{10}$$

Using transformation formulae of unit vectors given in [5] we have

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = -\mathrm{i}\hbar \boldsymbol{e}_z \Lambda \tag{11}$$

where

$$\Lambda = f\frac{\partial}{\partial\theta} + g\frac{\partial}{\partial\mu}. \tag{12}$$

It is straightforward to establish the following relations in which subscripts indicate differentiation:

$$
\begin{aligned}
&f_\mu + g_\theta = 0 \qquad f_\theta - g_\mu = 0\\
&f_{\theta\theta} + f_{\mu\mu} = 0 \qquad g_{\theta\theta} + g_{\mu\mu} = 0\\
&\left[\frac{1}{h}, \Lambda\right] = 4\frac{fg}{h} = -\frac{2}{h} f_\theta.
\end{aligned}
$$

One can verify that

$$[\Lambda, \Lambda] = 0.$$

The free Hamiltonian is

$$H_0 = \frac{\hbar^2}{ma^2}\frac{1}{h}\left[\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\theta^2}\right]$$

and one can check that

$$
\begin{aligned}
\frac{ma^2}{\hbar^2}[H_0, \Lambda] = {} & \left\{\frac{2}{h} f_\theta + \left[\frac{1}{h}, \Lambda\right]\right\}\frac{\partial^2}{\partial\theta^2} + \left\{\frac{2}{h} g_\mu + \left[\frac{1}{h}, \Lambda\right]\right\}\frac{\partial^2}{\partial\mu^2}\\
& + \frac{2}{h}(g_\theta + f_\mu)\frac{\partial}{\partial\theta}\frac{\partial}{\partial\mu} + \frac{1}{h}(f_{\theta\theta} + f_{\mu\mu})\frac{\partial}{\partial\theta} + \frac{1}{h}(g_{\mu\mu} + g_{\theta\theta})\frac{\partial}{\partial\theta}\\
= {} & 0.
\end{aligned}
$$

The Hamiltonian *with* the vector potential is the usual expression

$$H = \frac{\hbar^2}{2m}\left[\boldsymbol{\nabla} - \frac{\mathrm{i}e}{\hbar c}\boldsymbol{A}\right]^2. \tag{13}$$

In proceeding further the question of gauge transformations arises since one may wish to choose a particularly convenient form of $\boldsymbol{A}$, such as $\boldsymbol{A}$ parallel to $\boldsymbol{e}_\theta$, the choice discussed in [5].

3. Gauge transformations

The field $\boldsymbol{A}$ given above in (1) and (4) can be expressed in terms of elliptic coordinates. Then

$$\boldsymbol{A} = \frac{\sqrt{2}\Phi(-g\boldsymbol{e}_\mu + f\boldsymbol{e}_\theta)}{\pi a h^{1/2}}. \tag{14}$$

The regular gauge transformation of [5] removes the component along $\boldsymbol{e}_\mu$ giving the new field along the tangent to the ellipse, i.e.

$$\boldsymbol{A}' = \frac{\sqrt{2}\Phi}{\pi a}.\frac{1}{h^{1/2}}\boldsymbol{e}_\theta. \tag{15}$$

With a singular gauge transformation one can totally remove $\boldsymbol{A}$ or $\boldsymbol{A}'$ from H in which case the Schrödinger operator becomes easily separable as in the derivation of Mathieu equations (of course, the wavefunctions have to be transformed accordingly in order to maintain the gauge covariance of the Schrödinger equation). In order to find these gauge transformations we proceed in analogy to one-flux line considerations [3]. We consider

$$
\begin{aligned}
\phi_1 + \phi_2 &= \tan^{-1}\left[\frac{y}{x-a}\right] + \tan^{-1}\left[\frac{y}{x+a}\right]\\
&= \tan^{-1}\left[\frac{\sinh\mu\sin\theta}{\cosh\mu\cos\theta - 1}\right] + \tan^{-1}\left[\frac{\sinh\mu\sin\theta}{\cosh\mu\cos\theta + 1}\right]
\end{aligned}
$$

from which we deduce

$$U_S^{(1)} := \mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)} = \frac{\sinh(\mu+\mathrm{i}\theta)}{\sinh(\mu-\mathrm{i}\theta)} = 2\frac{\sinh^2(\mu+\mathrm{i}\theta)}{h}. \tag{16}$$

We set

$$\boldsymbol{A}_S = \boldsymbol{A} - \boldsymbol{\nabla}\Omega \qquad \Omega := \frac{\Phi}{2\pi}(\phi_1+\phi_2)$$

and verify that $\boldsymbol{A}_S = 0$. In fact,

$$\boldsymbol{\nabla}\Omega = \frac{\Phi}{2\pi}\sqrt{\frac{2}{h}}\left(\boldsymbol{e}_\mu\frac{\partial}{\partial\mu}, \boldsymbol{e}_\theta\frac{\partial}{\partial\theta}\right)\ln\left[\frac{\sinh(\mu+\mathrm{i}\theta)}{\sinh(\mu-\mathrm{i}\theta)}\right] = \boldsymbol{A}$$

and $\boldsymbol{A}$ is seen to be a pure gauge potential (apart from the cut).

The singular gauge transformation $U_S^{(1)}$ from $\boldsymbol{A}$ to 0 does not allow a separation of μ and θ as we see from the above. However, the singular gauge transformation from $\boldsymbol{A}'$ to 0 is easy, i.e.

$$U = \mathrm{e}^{+\mathrm{i}\frac{e}{\hbar c}\Omega} \tag{17}$$

with

$$\Omega = \frac{\Phi}{\pi}\theta \tag{18}$$

since then

$$\boldsymbol{\nabla}\Omega = \boldsymbol{\nabla}_\theta\Omega \equiv \frac{\sqrt{2}\boldsymbol{e}_\theta}{ah^{1/2}}\frac{\partial}{\partial\theta}\Omega \tag{19}$$

cancels exactly $\boldsymbol{A}'$. The singular behaviour results from the multivaluedness of $\theta = \tan^{-1}\left[\frac{y\cosh\mu}{x\sinh\mu}\right]$ and $\boldsymbol{A}_S$ is again (almost) pure gauge. Next we examine the transformation of L and H under these gauge transformations. Considering $L = \boldsymbol{e}_z\cdot(\boldsymbol{r}\times\boldsymbol{p})$ with $\boldsymbol{p} = -\mathrm{i}\hbar\boldsymbol{\nabla}$ we have

$$U^+LU = L + \frac{e}{c}\frac{\Phi}{\pi}f. \tag{20}$$

In the limit $\mu\to\infty, a\to 0$ this expression can be seen to reduce to the corresponding result of the one-solenoid case considered in [3].

We now examine the transformation of L under the singular gauge transformation $U_S^{(1)}$. A lengthy calculation yields

$$U_S^{+(1)}LU_S^{(1)} = L - 2\hbar(f^2-g^2) = -\hbar\left\{\frac{U_+}{R_+} + \frac{U_-}{R_-}\right\}. \tag{21}$$

Inserting constants we set

$$U_S := \mathrm{e}^{[\mathrm{i}(\phi_1+\phi_2)\alpha]} \qquad \text{with } \alpha = \frac{e}{\hbar c}\frac{\Phi}{2\pi}$$

and so

$$U_S^+LU_S = L - \frac{e}{c}\frac{\Phi}{2\pi}\left\{\frac{U_+}{R_+} + \frac{U_-}{R_-}\right\}. \tag{22}$$

Thus, in view of the violation of spherical symmetry this expression again does not agree with the conserved quantity.

Next we consider the Hamiltonian H. Since the Lagrangian is not rotationally symmetric the Hamiltonian H is also not rotationally symmetric. This may be verified explicitly by computing, e.g. the part $[\Lambda, \boldsymbol{A}^2]$ which will be found to be nonzero. However, with the singular gauge transformation we can—effectively—remove the vector potential from H and obtain H_0 provided the wavefunctions are transformed accordingly. The Schrödinger eigenvalue problem $H\psi = E\psi$ is then that of the two-dimensional Laplace operator separated in elliptic coordinates. This separation leads to two equations—a (periodic) Mathieu equation and a (nonperiodic) modified Mathieu equation, as is well known. Thus, with U

$$H \to U^+ H U = \tilde{H} \qquad \psi \to U^+\psi = \tilde{\psi}$$

and $\tilde{H}\tilde{\psi} = E\tilde{\psi}$ with $\tilde{\psi} = u(\theta)v(\mu)$ and $q = \frac{a^2 mE}{2\hbar^2}$ the Schrödinger equation separates into

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\mu^2} + (-\lambda + 2q\cosh 2\mu)v = 0 \tag{23}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + (\lambda - 2q\cos 2\theta)u = 0 \tag{24}$$

where λ is the separation constant. In the following we consider the eigenvalues of L and $\tilde{H}$.

4. Eigenvalues and eigenfunctions

We have seen that the angular momentum operator L commutes with the free part of the Hamiltonian, and we have also seen that the field $\boldsymbol{A}'$ can—effectively—be gauged away to give the Hamiltonian $\tilde{H}$ which is the free part of the original Hamiltonian expressed in elliptic coordinates. Thus

$$[L, \tilde{H}] = 0.$$

Hence L and $\tilde{H}$ share the same system of eigenfunctions $\tilde{\psi} = u(\theta)v(\mu)$. The angular part $u(\theta)$ is a solution of the periodic Mathieu equation obtained above. The solutions of this equation have been studied in great detail in the literature [7].

The question arises: What is the relation between L and M, where M is the classically conserved quantity given by equation (8)? Like L this quantity, i.e.

$$M = L - \frac{ae}{c}\frac{\Phi}{2\pi}\left[\frac{x-a}{R_-} - \frac{x+a}{R_+}\right]$$

does not commute with H and hence is not a conserved quantity in quantum mechanics. In comparing L and M we have to use the same gauge for both. M, by construction, is given in the original gauge, i.e. with $\boldsymbol{A}$ given by equation (4) or equation (14). $\tilde{H}$ is the Hamiltonian with $\boldsymbol{A} = 0$ and $[L, \tilde{H}] = 0$. To return to the original gauge we have to use the gauge transformation $U_S^{(1)}$ backwards, i.e.

$$U_S^{(1)}\tilde{H}U_S^{(1)+} = H$$

and (cf (21))

$$L' := U_S^{(1)} L U_S^{(1)+} = L + 2\hbar(f^2 - g^2).$$

Furthermore

$$\begin{aligned}[L', H] &= [U_S^{(1)} L U_S^{(1)+}, U_S^{(1)}\tilde{H}U_S^{(1)+}] \\ &= U_S^{(1)}\left[L, \tilde{H}\right]U_S^{(1)+} \\ &= 0.\end{aligned}$$

Thus, L' is a conserved quantity in the original gauge although H is not rotationally invariant in that gauge. Also $L' \neq M$.

We now return to the consideration of the solutions of Mathieu's equation referred to above. The solutions of the periodic Mathieu equation are characterized by the real Floquet parameter ν which enters in view of the Bloch function property [7]

$$u(\theta + \pi) = \mathrm{e}^{\mathrm{i}\nu\pi} u(\theta)$$

of the solutions of the Mathieu equation. Our original question was: What is the significance of ν, if any? Is it integral or nonintegral? In [6] it is claimed that ν is nonintegral—whereas [5] assumes without explanation that ν is integral. In order to clarify this point we look at the angular momentum operator L (which was not considered in [5, 6]) and at the solutions of the Mathieu equation in more detail.

The periodic solutions of the Mathieu equation in the neighbourhood of $q = 0$ (i.e. close to vanishing separation of the solenoids) are [7, 8]:

(i) for ν nonintegral

$$u \to me_\nu(\theta, q) = \mathrm{e}^{\mathrm{i}\nu\theta} + 0(q)$$

(with suitable normalization) and

$$\lambda = \nu^2 + 0(q) \tag{25}$$

and

(ii) for ν integral:

$$\nu \to 2n, 2n+1, 2n+1, 2n+2 \qquad \text{with } n = 0, 1, 2, \ldots$$

$$u \to \begin{cases} ce_{2n}(\theta, q) = \cos 2n\theta + 0(q) \\ ce_{2n+1}(\theta, q) = \cos(2n+1)\theta + 0(q) \\ se_{2n+1}(\theta, q) = \sin(2n+1)\theta + 0(q) \\ se_{2n+2}(\theta, q) = \sin(2n+2)\theta + 0(q) \end{cases}$$

(with suitable normalization) and respectively

$$\lambda \to \begin{cases} a_{2n}(q) = (2n)^2 + 0(q^2) = a_{2n}(-q) \\ a_{2n+1}(q) = (2n+1)^2 + 0(q) = b_{2n+1}(-q) \\ b_{2n+1}(q) = (2n+1)^2 + 0(q) = a_{2n+1}(-q) \\ b_{2n+2}(q) = (2n+2)^2 + 0(q^2) = b_{2n+2}(-q). \end{cases}$$

These solutions possess the following properties respectively

$$\begin{aligned} u(\theta + \pi) &= u(\theta) \\ u(\theta + \pi) &= -u(\theta) \\ u(\theta + \pi) &= -u(\theta) \\ u(\theta + \pi) &= u(\theta) \end{aligned}$$

and hence satisfy the relation $u(\theta + 2\pi) = u(\theta)$. We see that in the limit $q \to 0$ there is a confluence of the eigenvalues of (in each case) two distinct solutions, and the linear combination of these solutions which has modulus one is $\mathrm{e}^{\mathrm{i}N\theta}$, with N integral, and then agrees with the circular case and

$$\lim_{\substack{\nu \to N \\ q \to 0}} me_\nu(\theta, q) = \mathrm{e}^{\mathrm{i}N\theta}.$$

If ν was allowed to be nonintegral in general (in our elliptic case) we would have a multivalued wavefunction in the limiting case of $q \to 0$, in contradiction with elementary experience and the consequent requirement of single-valuedness upon replacement of θ by $\theta + 2\pi$ [3]. Thus, in order to retain this limiting property ν must be integral. In fact, this is also seen to be clear by looking at L in the limit $a \to 0, \mu \to \infty$ with $\frac{1}{2}a\mathrm{e}^{\mu} = r =$ finite. In this case $f \to 1, g \to 0$ and $L \to -\mathrm{i}\hbar\frac{\partial}{\partial\theta}$. The singular gauge transformation does not enter in this part of the discussion at all—only the fact that L commutes with the two-dimensional Laplacian separated in elliptic coordinates. As emphasized in [3] the eigenvalues of the angular momentum remain the same integers irrespective of whether a solenoid is present or not.

We now consider the eigenfunctions of H. Again we demand that in the limit $a \to 0, \mu \to \infty$ with $\frac{1}{2}a\mathrm{e}^{\mu} = r$ finite the findings of the one-solenoid case must be reproduced. The case of the latter has been investigated thoroughly in [3]. By allowing Φ to have some r dependence and Ω of equation (18) to be suitably regularized, the singular gauge transformation becomes nonsingular and, in fact, the gauge transformed field of $\boldsymbol{A}'$ (in fact, the r-component) is no longer zero but a rotationally noncovariant quantity proportional to the angle θ and $\frac{\mathrm{d}\Phi}{\mathrm{d}r}$. Thus, after one rotation the field assumes a new value. If the field changes like this, there is no reason why the wavefunction ψ of the corresponding Schrödinger equation $H\psi = E\psi$ should not change and be single-valued. It is perfectly acceptable in this case that the wavefunction ψ is multivalued. Thus, there is no reason why the wavefunction in the singular gauge and for $q \neq 0$ should have a very different behaviour. The multivaluedness of

$$\psi = U^{+}u(\theta)v(\mu) \tag{26}$$

as a result of the factor U^{+} is therefore perfectly acceptable.

5. Conclusions

We have demonstrated that the conserved angular momentum is quantized in integral units of $\hbar$, and that this is equal to the kinematical orbital momentum shifted by the flux. We have also demonstrated that one cannot *a priori* assume that the Schrödinger wavefunction, even after regularization of the singular gauge, is single-valued upon rotation by 2π. It is necessary to consider also the angular momentum operator whose eigenfunctions are not affected by the gauge transformation. This consideration then leads to integral values of the Floquet parameter which (multiplied by $\hbar$) give the eigenvalues of angular momentum. As described above, the use of the singular gauge transformation has the advantage of removing the electromagnetic field from the Hamiltonian of the Schrödinger equation, thus permitting the latter's separation into periodic and modified Mathieu equations. The singular gauge transformation implies the multivaluedness of the gauge function. A regularization of the latter implies a new gauge field which is no longer zero and hence a modification of the wavefunction ψ which therefore need no longer be single-valued. Thus, the considerations here—motivated by separability into Mathieu equations—rely on the use of a singular gauge transformation. This is the opposite approach to that of [9] which proves the single-valuedness of wavefunction as a consequence of several nonsingular gauges which are matched in regions of overlap. Other aspects of the two-solenoid problem, such as the calculation of the Green's function, have been investigated in [10] using a universal covering space technique based on the idea of multiply connected spaces associated with multivalued wavefunctions.

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References

[1] Wilczek F 1982 *Phys. Rev. Lett.* **48** 1144
Wilczek F 1982 *Phys. Rev. Lett.* **48** 1146
Wilczek F 1982 *Phys. Rev. Lett.* **49** 957
[2] Arovas D P, Schrieffer R, Wilczek F and Zee A 1985 *Nucl. Phys.* B **251** 117
[3] Jackiw R and Redlich A N 1983 *Phys. Rev. Lett.* **50** 555
[4] Jackiw R and Rebbi C 1976 *Phys. Rev. Lett.* **36** 1116
[5] Gu Z-Y and Qian S-W 1988 *J. Phys. A: Math. Gen.* **21** 2573
[6] de Veigy A D and Ouvry S 1993 *Phys. Lett.* B **307** 91
[7] Meixner J and Schäfke F W 1954 *Mathieusche Funktionen und Sphäroidfunktionen* (Berlin: Springer)
Arscott F M 1964 *Periodic Differential Equations* (London: Pergamon)
[8] Aly H H, Müller–Kirsten H J W and Vahedi-Faridi N 1975 *J. Math. Phys.* **16** 96
[9] Bawin M and Burnel A 1985 *J. Phys. A: Math. Gen.* **18** 2123
[10] Stovicek P 1989 *Phys. Lett.* A **142** 5